

THE S_4 -ACTION ON THE TETRAHEDRON ALGEBRA

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ABSTRACT. The action of the symmetric group S_4 on the Tetrahedron algebra, introduced by Hartwig and Terwilliger [HT05], is studied. This action gives a grading of the algebra which is related to its decomposition in [HT05] into a direct sum of three subalgebras isomorphic to the Onsager algebra. The ideals of both the Tetrahedron algebra and the Onsager algebra are determined.

INTRODUCTION

The Tetrahedron algebra \mathfrak{g}_{\boxtimes} has been defined in [HT05, Definition 1.1], in connection with the so called Onsager algebra introduced in [Ons44], in which the free energy of the two dimensional Ising model was computed. Since then it has been investigated by physicists and mathematicians in connection with solvable lattice models, representation theory, Kac-Moody Lie algebras, tridiagonal pairs and partially orthogonal polynomials (see [HT05] and the references there in).

One of the main results (Theorem 11.5) in [HT05] shows that \mathfrak{g}_{\boxtimes} is isomorphic to the three point \mathfrak{sl}_2 loop algebra $\mathfrak{sl}_2 \otimes \mathcal{A}$, where \mathcal{A} is the algebra $k[t, t^{-1}, (1-t)^{-1}]$ (a subalgebra of the field of rational functions on the indeterminate t).

The Tetrahedron algebra is endowed with an action of the symmetric group S_4 by automorphisms, and the result above is used in [HT05, Theorem 11.6] to show that \mathfrak{g}_{\boxtimes} is the direct sum $\mathfrak{g}_{\boxtimes} = \Omega \oplus \Omega' \oplus \Omega''$ of a subalgebra Ω , which is isomorphic to the Onsager algebra, and its images under the action of the cycle (123). Hence, \mathfrak{g}_{\boxtimes} is a direct sum of three subalgebras (not ideals!) which are isomorphic to the Onsager algebra.

On the other hand, like any other Lie algebra endowed with an action of S_4 by automorphisms, \mathfrak{g}_{\boxtimes} is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded by the action of Klein's 4 group (see [EO05]):

$$\mathfrak{g}_{\boxtimes} = (\mathfrak{g}_{\boxtimes})_{(\bar{0}, \bar{0})} \oplus (\mathfrak{g}_{\boxtimes})_{(\bar{1}, \bar{0})} \oplus (\mathfrak{g}_{\boxtimes})_{(\bar{0}, \bar{1})} \oplus (\mathfrak{g}_{\boxtimes})_{(\bar{1}, \bar{1})}. \quad (0.1)$$

In [HT05, Problem 13.4], the authors pose the problem of showing that $(\mathfrak{g}_{\boxtimes})_{(\bar{0}, \bar{0})} = 0$, of finding a basis for each of the subspaces $(\mathfrak{g}_{\boxtimes})_{(\bar{1}, \bar{0})}$, $(\mathfrak{g}_{\boxtimes})_{(\bar{0}, \bar{1})}$ and $(\mathfrak{g}_{\boxtimes})_{(\bar{1}, \bar{1})}$ (which become abelian subalgebras), and of relating this $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading to the previous decomposition $\mathfrak{g}_{\boxtimes} = \Omega \oplus \Omega' \oplus \Omega''$.

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These notes give a solution to this problem. A key to this solution will be the use of a very suitable basis of $\mathfrak{sl}_2 \otimes \mathcal{A}$, as a Lie algebra over \mathcal{A} , which is also useful in simplifying some of the arguments in [HT05].

All the algebras considered will be defined over a ground field k of characteristic $\neq 2$. Unadorned tensor products \otimes will be considered over k .

In the next section, the isomorphism between \mathfrak{g}_{\boxtimes} and $\mathfrak{sl}_2 \otimes \mathcal{A}$, as algebras with an action of the symmetric group S_4 , given in [HT05], will be reviewed and proved in a simplified way, by using the suitable \mathcal{A} -basis mentioned above. The action of S_4 on \mathfrak{g}_{\boxtimes} translates into an action of Klein's 4 group as \mathcal{A} -automorphisms of $\mathfrak{sl}_2 \otimes \mathcal{A}$ plus an action of S_3 on both \mathfrak{sl}_2 and \mathcal{A} . Section 2 will be devoted to solve [HT05, Problem 13.4], and the normal Lie related triple algebra which is associated to the action of S_4 on \mathfrak{g}_{\boxtimes} , as shown in [EO05], will be found in Section 3. This will highlight a general construction of normal Lie related triple algebras defined on certain commutative associative algebras endowed with an action of the symmetric group S_3 . Section 4 will be devoted to solve [HT05, Problem 13.3], which asks for the ideals of the Tetrahedron algebra, in terms of this \mathcal{A} -basis used throughout. A general result on ideals of some Lie algebras will be given, and the ideals of the Onsager algebra will be determined too. Section 5 will give a different presentation of the Tetrahedron algebra by generators and relations, inspired in the properties of the \mathcal{A} -basis used throughout. Finally, Section 6 will deal with the universal central extension of the Tetrahedron algebra studied in [BT].

1. THE TETRAHEDRON ALGEBRA AND THE THREE POINT \mathfrak{sl}_2 LOOP ALGEBRA

The Tetrahedron algebra \mathfrak{g}_{\boxtimes} has been defined in [HT05]. It is the Lie algebra over k with generators

$$\{X_{ij} : i, j \in \{0, 1, 2, 3\}, i \neq j\} \quad (1.1)$$

and the relations

$$X_{ij} + X_{ji} = 0 \quad \text{for } i \neq j, \quad (1.2a)$$

$$[X_{ij}, X_{jk}] = 2(X_{ij} + X_{jk}) \quad \text{for mutually distinct } i, j, k, \quad (1.2b)$$

$$[X_{hi}, [X_{hi}, [X_{hi}, X_{jk}]]] = 4[X_{hi}, X_{jk}] \quad \text{for mutually distinct } h, i, j, k. \quad (1.2c)$$

One of the main results in [HT05] relates the Tetrahedron algebra to the three point \mathfrak{sl}_2 loop algebra $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$, where \mathfrak{sl}_2 is the Lie algebra of two by two traceless matrices over k and \mathcal{A} is the unital commutative associative algebra $k[t, t^{-1}, (1-t)^{-1}]$ (t an indeterminate), which is a subalgebra of the field of fractions $k(t)$. To present the precise relationship in [HT05, Proposition 6.5 and Theorem 11.5], first consider the basis $\{x = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$ of \mathfrak{sl}_2 , whose elements satisfy $[x, y] = 2(x + y)$, $[y, z] = 2(y + z)$, $[z, x] = 2(z + x)$. Then, because of its own definition by generators and relations, there is a Lie algebra homomorphism [HT05, Proposition 6.5]

$$\Psi : \mathfrak{g}_{\boxtimes} \longrightarrow \mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$$

determined by:

$$\begin{aligned}\Psi(X_{12}) &= x \otimes 1, & \Psi(X_{23}) &= y \otimes 1, & \Psi(X_{31}) &= z \otimes 1, \\ \Psi(X_{03}) &= y \otimes t + z \otimes (t - 1), \\ \Psi(X_{01}) &= z \otimes t' + x \otimes (t' - 1), \\ \Psi(X_{02}) &= x \otimes t'' + y \otimes (t'' - 1),\end{aligned}\tag{1.3}$$

where $t' = 1 - t^{-1}$ and $t'' = (1 - t)^{-1}$ (see [HT05, Lemma 6.2]). This homomorphism Ψ was proved to be an isomorphism in [HT05, Theorem 11.5]. Here this will be proved in another way. Some useful results will come out during this process.

The symmetric group

$$S_4 = \{\sigma : \{0, 1, 2, 3\} \rightarrow \{0, 1, 2, 3\} : \sigma \text{ is a bijective map}\}$$

embeds naturally in the group of automorphisms $\text{Aut}(\mathfrak{g}_{\boxtimes})$ by means of

$$\sigma(X_{ij}) = X_{\sigma(i)\sigma(j)},$$

for any $\sigma \in S_4$ and $0 \leq i \neq j \leq 3$. (Here the actions will always be taken on the left.)

Consider the following generators of S_4 :

$$\begin{aligned}\tau_1 &= (12)(30) : 1 \leftrightarrow 2, 3 \leftrightarrow 0, \\ \tau_2 &= (23)(10) : 2 \leftrightarrow 3, 1 \leftrightarrow 0, \\ \varphi &= (123) : 1 \mapsto 2 \mapsto 3 \mapsto 1, 0 \mapsto 0, \\ \tau &= (12) : 1 \leftrightarrow 2, 0 \mapsto 0, 3 \mapsto 3.\end{aligned}$$

The elements τ_1 and τ_2 generate Klein's 4 group, while φ and τ generate a copy of the symmetric group S_3 (recall that S_4 is the semidirect product of these two subgroups).

Theorem 1.4. *S_4 embeds as a group of automorphisms of $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$ in the following way:*

- (i) $\varphi = \varphi_{\mathfrak{s}} \otimes \varphi_{\mathcal{A}}$, where $\varphi_{\mathfrak{s}}$ is the order 3 automorphism of \mathfrak{sl}_2 given by

$$\varphi_{\mathfrak{s}}(x) = y, \quad \varphi_{\mathfrak{s}}(y) = z, \quad \varphi_{\mathfrak{s}}(z) = x,$$

and $\varphi_{\mathcal{A}}$ is the order 3 automorphism of the k -algebra \mathcal{A} determined by

$$\varphi_{\mathcal{A}}(t) = 1 - t^{-1} = t'.$$

In particular, φ is an \mathcal{A} -semilinear automorphism of \mathfrak{g} and $\varphi_{\mathcal{A}}$ is its associated automorphism of \mathcal{A} . That is, $\varphi(ga) = \varphi(g)\varphi_{\mathcal{A}}(a)$ for any $g \in \mathfrak{g}$ and $a \in \mathcal{A}$.

- (ii) $\tau = \tau_{\mathfrak{s}} \otimes \tau_{\mathcal{A}}$, where $\tau_{\mathfrak{s}}$ is the order 2 automorphism of \mathfrak{sl}_2 given by

$$\tau_{\mathfrak{s}}(x) = -x, \quad \tau_{\mathfrak{s}}(y) = -z, \quad \tau_{\mathfrak{s}}(z) = -y,$$

and $\tau_{\mathcal{A}}$ is the order 2 automorphism of \mathcal{A} determined by $\tau_{\mathcal{A}}(t) = 1 - t$. In particular, τ is an \mathcal{A} -semilinear automorphism of \mathfrak{g} and $\tau_{\mathcal{A}}$ is its associated automorphism of \mathcal{A} .

(iii) τ_1 is the automorphism of \mathfrak{g} , as a Lie algebra over \mathcal{A} , given by

$$\begin{aligned}\tau_1(x \otimes 1) &= -x \otimes 1, \\ \tau_1(y \otimes 1) &= -(z \otimes t' + x \otimes (t' - 1)), \\ \tau_1(z \otimes 1) &= x \otimes t'' + y \otimes (t'' - 1).\end{aligned}\tag{1.5}$$

(iv) τ_2 is the automorphism of \mathfrak{g} , as a Lie algebra over \mathcal{A} , given by

$$\begin{aligned}\tau_2(x \otimes 1) &= y \otimes t + z \otimes (t - 1), \\ \tau_2(y \otimes 1) &= -y \otimes 1, \\ \tau_2(z \otimes 1) &= -(x \otimes t'' + y \otimes (t'' - 1)).\end{aligned}\tag{1.6}$$

Moreover, under these actions of S_4 on \mathfrak{g}_{\boxtimes} and on \mathfrak{g} , the homomorphism Ψ in (1.3) becomes a homomorphism of Lie algebras with S_4 -action. That is,

$$\Psi(\sigma(X)) = \sigma(\Psi(X)),\tag{1.7}$$

for any $\sigma \in S_4$ and $X \in \mathfrak{g}_{\boxtimes}$.

Proof. It is easy to check that $\varphi_{\mathfrak{s}}$ and $\tau_{\mathfrak{s}}$ are automorphisms of \mathfrak{sl}_2 of order 3 and 2, respectively, and that $\varphi_{\mathfrak{s}}\tau_{\mathfrak{s}} = \tau_{\mathfrak{s}}\varphi_{\mathfrak{s}}^2$. Also, $\varphi_{\mathcal{A}}\tau_{\mathcal{A}} = \tau_{\mathcal{A}}\varphi_{\mathcal{A}}^2$. Hence S_3 embeds in $\text{Aut } \mathfrak{g}$ by identifying φ to $\varphi_{\mathfrak{s}} \otimes \varphi_{\mathcal{A}}$ and τ to $\tau_{\mathfrak{s}} \otimes \tau_{\mathcal{A}}$.

Now, equation (1.5) defines a unique Lie algebra homomorphism of \mathfrak{g} , as a Lie algebra over \mathcal{A} , which satisfies $\tau_1^2 = 1$. The same happens for τ_2 and $\tau_1\tau_2 = \tau_2\tau_1$. For instance, using that $(t' - 1)t = -1 = t''(t - 1)$, we check

$$\begin{aligned}\tau_1\tau_2(x \otimes 1) &= \tau_1(y \otimes t + z \otimes (t - 1)) \\ &= -z \otimes t't - x \otimes (t' - 1)t + x \otimes t''(t - 1) + y \otimes (t'' - 1)(t - 1) \\ &= -z \otimes (t - 1) + x \otimes 1 - x \otimes 1 - y \otimes t \\ &= -(y \otimes t + z \otimes (t - 1)) \\ &= -\tau_2(x \otimes 1) = \tau_2\tau_1(x \otimes 1).\end{aligned}$$

Now, it has to be checked that $\varphi\tau_1 = \tau_2\varphi$, $\varphi\tau_2 = \tau_1\tau_2\varphi$, $\tau\tau_1 = \tau_1\tau$ and $\tau\tau_2 = \tau_1\tau_2\tau$. In all cases, the maps on both sides are \mathcal{A} -semilinear with the same associated automorphisms of \mathcal{A} , so it is enough to check the equalities on the basis $\{x \otimes 1, y \otimes 1, z \otimes 1\}$ of \mathfrak{g} over \mathcal{A} . This is straightforward.

For the last part, it is enough to check (1.7) for $X = X_{ij}$, $i \neq j$ in $\{0, 1, 2, 3\}$, and again this is a routine verification. \square

In order to show that Ψ is an isomorphism, it is better to work with a different basis of \mathfrak{g} over \mathcal{A} . Consider the following elements of \mathfrak{g} :

$$\begin{aligned}u_0 &= \frac{1}{4}\Psi(X_{02} + X_{31}) = \frac{1}{4}(z \otimes 1 + x \otimes t'' + y \otimes (t'' - 1)), \\ u_1 &= \frac{1}{4}\Psi(X_{03} + X_{12}) = \frac{1}{4}(x \otimes 1 + y \otimes t + z \otimes (t - 1)), \\ u_2 &= \frac{1}{4}\Psi(X_{01} + X_{23}) = \frac{1}{4}(y \otimes 1 + z \otimes t' + x \otimes (t' - 1)).\end{aligned}\tag{1.8}$$

Observe that, since $\varphi_{\mathcal{A}}(t) = t'$ and $\varphi_{\mathcal{A}}(t') = t''$, these elements are permuted cyclically by the order 3 automorphism φ in Theorem 1.4.

Theorem 1.9. *With u_0 , u_1 and u_2 as above:*

- (i) $\{u_0, u_1, u_2\}$ is a basis of \mathfrak{g} as a module over \mathcal{A} .

$$(ii) \quad [u_0, u_1] = -u_2t, \quad [u_1, u_2] = -u_0t', \quad [u_2, u_0] = -u_1t''.$$

(iii) u_0, u_1 and u_2 generate \mathfrak{g} as a Lie algebra over k .

Proof. Let us start with item (ii). Use Ψ and the relation (1.2b) to get

$$\begin{aligned} [u_0, u_1] &= \frac{1}{16} \Psi([X_{02} + X_{31}, X_{03} + X_{12}]) \\ &= \frac{1}{16} \Psi(-[X_{20}, X_{03}] - [X_{02}, X_{21}] + [X_{13}, X_{30}] + [X_{31}, X_{12}]) \\ &= \frac{1}{16} \Psi(-2(X_{20} + X_{03}) - 2(X_{02} + X_{21}) \\ &\quad + 2(X_{13} + X_{30}) + 2(X_{31} + X_{12})) \\ &= \frac{1}{16} \Psi(-4X_{03} + 4X_{12}) = -\frac{1}{4} \Psi(X_{03} - X_{12}) \\ &= -\frac{1}{4} \Psi(-x \otimes 1 + y \otimes t + z \otimes (t-1)). \end{aligned}$$

On the other hand,

$$\begin{aligned} -u_2t &= -\frac{1}{4}(y \otimes t + z \otimes t't + x \otimes (t'-1)t) \\ &= -\frac{1}{4}(y \otimes t + z \otimes (t-1) - x \otimes 1), \end{aligned}$$

since $(t'-1)t = -1$. Hence $[u_0, u_1] = -u_2t$. Now apply φ to get $[u_1, u_2] = [\varphi(u_0), \varphi(u_1)] = -\varphi(u_2t) = -\varphi(u_2)\varphi_{\mathcal{A}}(t) = -u_0t'$ and, in the same vein, $[u_2, u_0] = -u_1t''$, as required.

To prove (i), first note that $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$ is a subalgebra of the simple Lie algebra $\mathfrak{sl}_2 \otimes k(t)$ over the field $k(t)$. The $k(t)$ -subalgebra \mathfrak{s} generated by u_0, u_1 and u_2 is perfect ($\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$) by (ii), and hence its dimension cannot be ≤ 2 (otherwise, it would be a solvable Lie algebra). Therefore, $\{u_0, u_1, u_2\}$ is a basis of $\mathfrak{sl}_2 \otimes k(t)$ over $k(t)$. In particular, u_0, u_1 and u_2 are linearly independent over \mathcal{A} , and $u_0\mathcal{A} \oplus u_1\mathcal{A} \oplus u_2\mathcal{A}$ is an \mathcal{A} -subalgebra of \mathfrak{g} . Moreover, the computations above show that

$$4u_2t = y \otimes t + z \otimes (t-1) - x \otimes 1, \quad (1.10)$$

while $4u_1 = x \otimes 1 + y \otimes t + z \otimes (t-1)$. Hence,

$$x \otimes 1 = 2(u_1 - u_2t) \quad (1.11)$$

belongs to $u_0\mathcal{A} \oplus u_1\mathcal{A} \oplus u_2\mathcal{A}$. Apply φ to get $y \otimes 1 = 2(u_2 - u_0t') \in u_0\mathcal{A} \oplus u_1\mathcal{A} \oplus u_2\mathcal{A}$ and also $z \otimes 1 = 2(u_0 - u_1t'') \in u_0\mathcal{A} \oplus u_1\mathcal{A} \oplus u_2\mathcal{A}$. This proves (i).

Finally, let us denote now by \mathfrak{s} the k -subalgebra of \mathfrak{g} generated by u_0, u_1 and u_2 . Observe that \mathfrak{s} is invariant under the action of the order 3 automorphism φ , so it is enough to prove that $u_0\mathcal{A}$ is contained in \mathfrak{s} . From (ii) we obtain

$$\begin{aligned} [u_1, [u_1, u_0t^n]] &= [u_1, u_2t^{n+1}] = -u_0t't^{n+1} \\ &= -u_0t^n(t-1) = -u_0t^{n+1} + u_0t^n \end{aligned}$$

since $t't = t - 1$. Hence, an induction argument shows that $u_0 t^n$ is in \mathfrak{s} for any $n \geq 0$. In the same vein,

$$[u_2, [u_2, u_0(t')^n]] = [u_2, u_1(t')^n t''] = -u_0(t')^{n+1} t'' = -u_0(t')^n (t' - 1),$$

and $u_0(t')^n \in \mathfrak{s}$ for any $n \geq 0$. Finally, since $u_0 t^n$ is in \mathfrak{s} for any $n \geq 0$, and \mathfrak{s} is invariant under φ , we get $u_2(t'')^n \in \mathfrak{s}$ too, for any $n \geq 0$. Hence:

$$[u_1, u_2(t'')^n (1 - t'')] = -u_0(t'')^n (1 - t'') t' = -u_0(t'')^n \in \mathfrak{s}$$

for any $n \geq 0$. But $\{1\} \cup \{t^n, (t')^n, (t'')^n : n \in \mathbb{N}\}$ is a basis of \mathcal{A} over k ([HT05, Lema 6.3]), so $u_0 \mathcal{A}$ is contained in \mathfrak{s} , as required. \square

Corollary 1.12. *The homomorphism Ψ is onto.*

As in [HT05], let Ω (respectively Ω' , Ω'') denote the subalgebra of \mathfrak{g}_{\boxtimes} generated by X_{12} and X_{03} (respectively X_{23} and X_{01} , X_{31} and X_{02}). Note that $\Omega' = \varphi(\Omega)$ and $\Omega'' = \varphi(\Omega')$. In [HT05, Proposition 7.8] it is proved that \mathfrak{g}_{\boxtimes} is the direct sum of the subalgebras Ω , Ω' and Ω'' . A simpler proof can be given as follows:

Lemma 1.13. *Let S_1 and S_2 be two subspaces of a Lie algebra \mathfrak{l} such that $[S_1, S_2] \subseteq S_1 + S_2$ holds, and let \mathfrak{s}_i be the subalgebra generated by S_i , $i = 1, 2$. Then $[\mathfrak{s}_1, \mathfrak{s}_2] \subseteq \mathfrak{s}_1 + \mathfrak{s}_2$. In particular, $\mathfrak{s}_1 + \mathfrak{s}_2$ is a subalgebra of \mathfrak{l} .*

Proof. From $[S_1, S_2] \subseteq S_1 + S_2$, it follows that $[S_1, \mathfrak{s}_2] \subseteq S_1 + \mathfrak{s}_2$, and then that $[\mathfrak{s}_1, \mathfrak{s}_2] \subseteq \mathfrak{s}_1 + \mathfrak{s}_2$. \square

Proposition 1.14. $\mathfrak{g}_{\boxtimes} = \Omega + \Omega' + \Omega''$.

Proof. Let S (respectively S' , S'') denote the subspace spanned by X_{12} and X_{03} (respectively X_{23} and X_{01} , X_{31} and X_{02}). Then, by (1.2b), $[S, S'] \subseteq S + S'$, so the previous Lemma gives $[\Omega, \Omega'] \subseteq \Omega + \Omega'$, and, similarly, $[\Omega', \Omega''] \subseteq \Omega' + \Omega''$ and $[\Omega'', \Omega] \subseteq \Omega'' + \Omega$. Therefore, $\Omega + \Omega' + \Omega''$ is a subalgebra of \mathfrak{g}_{\boxtimes} , which contains all the generators X_{ij} , so it is the whole \mathfrak{g}_{\boxtimes} . \square

The images under Ψ of these subalgebras are given in the next result:

Proposition 1.15.

- (i) $\Psi(\Omega) = u_0(t-1)k[t] \oplus u_1k[t] \oplus u_2tk[t],$
- (ii) $\Psi(\Omega') = u_0t'k[t'] \oplus u_1(t'-1)k[t'] \oplus u_2k[t'],$
- (iii) $\Psi(\Omega'') = u_0k[t''] \oplus u_1t''k[t''] \oplus u_2(t''-1)k[t''].$

In particular, \mathfrak{g} is the direct sum of the subalgebras $\Psi(\Omega)$, $\Psi(\Omega')$, and $\Psi(\Omega'')$.

Proof. (ii) and (iii) are obtained from (i) by applying the order 3 automorphism φ . To prove (i), first note that since $tt' = t - 1$ and $t''(t-1) = -1$, the k -subspace $u_0(t-1)k[t] \oplus u_1k[t] \oplus u_2tk[t]$ is a k -subalgebra of \mathfrak{g} , and it contains $u_1 = \frac{1}{4}\Psi(X_{03} + X_{12})$ and $u_2t = \frac{1}{4}\Psi(X_{03} - X_{12})$ (1.10), so it certainly contains $\Psi(\Omega)$. But one has $\mathcal{A} = (t-1)k[t] \oplus t'k[t'] \oplus k[t'']$ because of [HT05, Lemma 6.3], so applying $\varphi_{\mathcal{A}}$, also $\mathcal{A} = k[t] \oplus (t'-1)k[t'] \oplus t''k[t''] = tk[t] \oplus k[t'] \oplus (t''-1)k[t'']$. Hence \mathfrak{g} is the direct sum of the subalgebras on the right hand sides of (i), (ii), and (iii). Since Ψ is onto (Corollary 1.12) and $\mathfrak{g}_{\boxtimes} = \Omega + \Omega' + \Omega''$ (Proposition 1.14), the result follows. \square

The next result simplifies the work to prove that Ψ is one-to-one:

Lemma 1.16. *If the restriction of Ψ to Ω is one-to-one, so is Ψ .*

Proof. Assume that the restriction of Ψ to Ω is one-to-one. Then, since $\Psi(\Omega \cap (\Omega' + \Omega''))$ is contained in $\Psi(\Omega) \cap (\Psi(\Omega') + \Psi(\Omega'')) = 0$ (as \mathfrak{g} is the direct sum of $\Psi(\Omega)$, $\Psi(\Omega')$, and $\Psi(\Omega'')$ by Proposition 1.15), it follows that $\Omega \cap (\Omega' + \Omega'')$ is contained in the kernel of the restriction $\Psi|_{\Omega}$, which is assumed to be 0. Hence, by the cyclic symmetry provided by φ , $\mathfrak{g}_{\boxtimes} = \Omega \oplus \Omega' \oplus \Omega''$ and, again using the cyclic symmetry, the restriction of Ψ to each of these three direct summands is one-to-one, and so is Ψ . \square

Therefore, it is enough to prove that $\Psi|_{\Omega} : \Omega \rightarrow \mathfrak{g}$ is one-to-one. But Ω is the homomorphic image of the Onsager algebra \mathcal{O} (see [HT05, Section 4]), which is the Lie algebra over k with generators A and B and relations

$$[A, [A, [A, B]]] = 4[A, B], \quad [B, [B, [B, A]]] = 4[B, A],$$

under the homomorphism which takes A to X_{12} and B to X_{03} .

Hence, in order to prove that $\Psi|_{\Omega}$ is one-to-one, it is enough to prove the following:

Lemma 1.17. *The Lie algebra homomorphism $\phi : \mathcal{O} \rightarrow \mathfrak{g}$ determined by $\phi(A) = \Psi(X_{12})$ and $\phi(B) = \Psi(X_{03})$ is one-to-one.*

Proof. First note that $\phi(A) = \Psi(X_{12}) = x \otimes 1 = 2(u_1 - u_2 t)$ (1.11), and $\phi(B) = \Psi(X_{03}) = y \otimes t + z \otimes (t - 1) = 2(u_1 + u_2 t)$ (1.10). Besides [Ons44], \mathcal{O} has a basis $\{A_m : m \in \mathbb{Z}\} \cup \{G_l : l \in \mathbb{N}\}$, where $A_0 = A$, $A_1 = B$ and

$$[A_l, A_m] = 2G_{l-m} \ (l > m), \quad [G_l, A_m] = A_{m+l} - A_{m-l}, \quad [G_l, G_m] = 0.$$

Denote by \mathcal{O}_A (respectively \mathcal{O}_G) the linear span of $\{A_m : m \in \mathbb{Z}\}$ (respectively $\{G_l : l \in \mathbb{N}\}$). Then $\text{ad}_{G_1}|_{\mathcal{O}_A} : \mathcal{O}_A \rightarrow \mathcal{O}_A$ is one-to-one, and so is $\text{ad}_{A_0}|_{\mathcal{O}_G} : \mathcal{O}_G \rightarrow \mathcal{O}_A$. This shows that any nonzero ideal of \mathcal{O} intersects nontrivially \mathcal{O}_A . Also, $\{(\text{ad}_{G_1})^m(A_0), (\text{ad}_{G_1})^m(A_1) : m \geq 0\}$ is a basis of \mathcal{O}_A , and so is $\{(\text{ad}_{G_1})^m(A_0 + A_1), (\text{ad}_{G_1})^m(A_0 - A_1) : m \geq 0\}$.

Hence, in order to prove that ϕ is one-to-one, it is enough to prove that so is $\phi|_{\mathcal{O}_A}$ and, hence, to prove that $\{\phi((\text{ad}_{G_1})^m(A_0 + A_1)), \phi((\text{ad}_{G_1})^m(A_0 - A_1)) : m \geq 0\}$ is a linearly independent set in \mathfrak{g} . But,

$$\begin{aligned} \phi(G_1) &= \frac{1}{2}\phi([A_1, A_0]) = \frac{1}{2}[\Psi(X_{03}), \Psi(X_{12})] \\ &= 2[u_1 + u_2 t, u_1 - u_2 t] = -4[u_1, u_2 t] = 4u_0 t t' = 4u_0(t - 1), \end{aligned}$$

$$\phi(A_0 + A_1) = 2(u_1 - u_2 t) + 2(u_1 + u_2 t) = 4u_1,$$

$$\phi(A_0 - A_1) = 2(u_1 + u_2 t) - 2(u_1 - u_2 t) = 4u_2 t,$$

so we must check that $\{(\text{ad}_{u_0(t-1)})^m(u_1), (\text{ad}_{u_0(t-1)})^m(u_2 t) : m \geq 0\}$ is a linearly independent set. Now,

$$\begin{aligned} [u_0(t - 1), u_1] &= -u_2(t - 1), \\ [u_0(t - 1), u_2 t] &= u_1 t''(t - 1)t = -u_1 t, \end{aligned}$$

so

$$\begin{aligned} (\mathrm{ad}_{u_0(t-1)})^{2m}(u_1) &= u_1(t(t-1))^m, \\ (\mathrm{ad}_{u_0(t-1)})^{2m}(u_2t) &= (u_2t)(t(t-1))^m, \\ (\mathrm{ad}_{u_0(t-1)})^{2m+1}(u_1) &= -(u_2t)(t-1)(t(t-1))^m, \\ (\mathrm{ad}_{u_0(t-1)})^{2m+1}(u_2t) &= -u_1t(t(t-1))^m, \end{aligned}$$

and all these elements are linearly independent over k . \square

Corollary 1.18. (*See [HT05, Theorems 11.5 and 11.6, and Corollary 12.5].*)

- (i) $\Psi : \mathfrak{g}_{\boxtimes} \rightarrow \mathfrak{g}$ is an isomorphism.
- (ii) \mathfrak{g}_{\boxtimes} is the direct sum of its subalgebras Ω , Ω' and Ω'' .
- (iii) Ω is isomorphic to the Onsager algebra.

Proof. (i) follows from the results above. Then (ii) follows from Proposition 1.15, and (iii) follows because the epimorphism $\mathcal{O} \rightarrow \Omega$ such that $A \mapsto X_{12}$ and $B \mapsto X_{03}$ is one-to-one by Lemma 1.17. \square

2. THE SOLUTION TO [HT05, Problem 13.4]

The action of S_4 on \mathfrak{g}_{\boxtimes} (or on $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$) restricts to an action of Klein's 4 group, which gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading on \mathfrak{g} , as in [EO05, (1.1)]:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad (2.1)$$

where

$$\begin{aligned} \mathfrak{t} &= \{g \in \mathfrak{g} : \tau_1(g) = g, \tau_2(g) = g\} (= \mathfrak{g}_{(\bar{0}, \bar{0})}), \\ \mathfrak{g}_0 &= \{g \in \mathfrak{g} : \tau_1(g) = g, \tau_2(g) = -g\} (= \mathfrak{g}_{(\bar{1}, \bar{0})}), \\ \mathfrak{g}_1 &= \{g \in \mathfrak{g} : \tau_1(g) = -g, \tau_2(g) = g\} (= \mathfrak{g}_{(\bar{0}, \bar{1})}), \\ \mathfrak{g}_2 &= \{g \in \mathfrak{g} : \tau_1(g) = -g, \tau_2(g) = -g\} (= \mathfrak{g}_{(\bar{1}, \bar{1})}). \end{aligned}$$

In [HT05, Problem 13.4] it is posed the question of proving that $\mathfrak{t} = 0$, of obtaining a basis for each of these subspaces \mathfrak{g}_i , and of investigating the relationship between this decomposition (2.1) and the decomposition $\mathfrak{g}_{\boxtimes} = \Omega \oplus \Omega' \oplus \Omega''$ in Corollary 1.18.

The use of the \mathcal{A} -basis $\{u_0, u_1, u_2\}$ of \mathfrak{g} (Theorem 1.9) makes the determination of the subspaces in (2.1) quite easy:

Theorem 2.2. *With the previous notations,*

$$\mathfrak{t} = 0, \quad \mathfrak{g}_0 = u_0\mathcal{A}, \quad \mathfrak{g}_1 = u_1\mathcal{A}, \quad \mathfrak{g}_2 = u_2\mathcal{A}.$$

Proof. Recall that $\tau_1 = (12)(03)$ and $\tau_2 = (23)(10)$. Since $\tau_1(X_{02}) = X_{31}$ and $\tau_2(X_{02}) = X_{13} = -X_{31}$, it follows that $u_0 = \frac{1}{4}\Psi(X_{02} + X_{31})$ belongs to \mathfrak{g}_0 . But the automorphisms τ_1 and τ_2 are \mathcal{A} -linear (Proposition 1.4), so $u_0\mathcal{A}$ is contained in \mathfrak{g}_0 . In the same vein it is proved that $u_1\mathcal{A} \subseteq \mathfrak{g}_1$ and $u_2\mathcal{A} \subseteq \mathfrak{g}_2$; and Theorem 1.9 finishes the proof. \square

Since the set $\{1\} \cup \{t^n, (t')^n, (t'')^n : n \in \mathbb{N}\}$ is a k -basis of \mathcal{A} ([HT05, Lemma 6.3]), the following result, which solves part of [HT05, Problem 13.4], is clear:

Corollary 2.3. *For $i = 0, 1, 2$, the set $\{u_i\} \cup \{u_i t^n, u_i(t')^n, u_i(t'')^n : n \in \mathbb{N}\}$ is a k -basis of the space \mathfrak{g}_i .*

Actually, [HT05, Lemma 6.3] shows that $\mathcal{A} = (t-1)k[t] \oplus t'k[t'] \oplus k[t'']$, so

$$\begin{aligned} \mathfrak{g}_0 &= u_0(t-1)k[t] \oplus u_0 t'k[t'] \oplus u_0 k[t''] \\ &= (\Psi(\Omega) \cap \mathfrak{g}_0) \oplus (\Psi(\Omega') \cap \mathfrak{g}_0) \oplus (\Psi(\Omega'') \cap \mathfrak{g}_0), \end{aligned}$$

(see Proposition 1.15), and something similar holds for \mathfrak{g}_1 and \mathfrak{g}_2 . This gives the relationship between the decompositions $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and the decomposition $\mathfrak{g}_{\boxtimes} = \Omega \oplus \Omega' \oplus \Omega''$.

Remark 2.4. Also, since $t''(t-1) = -1$ and $(t''-1)(t-1) = -t$, it follows that

$$\begin{aligned} u_0(t-1)k[t] &= \left(z \otimes (t-1) + x \otimes t''(t-1) + y \otimes (t''-1)(t-1) \right) k[t] \\ &\subseteq x \otimes k[t] \oplus y \otimes tk[t] \oplus z \otimes (t-1)k[t]. \end{aligned}$$

Besides, the definition of u_1 shows immediately that

$$u_1 k[t] \subseteq x \otimes k[t] \oplus y \otimes tk[t] \oplus z \otimes (t-1)k[t],$$

and since $t't = t-1$ and $(t'-1)t = -1$, also

$$u_2 tk[t] \subseteq x \otimes k[t] \oplus y \otimes tk[t] \oplus z \otimes (t-1)k[t].$$

In this way, one recovers [HT05, Corollary 13], which asserts that

$$\Psi(\Omega) = x \otimes k[t] \oplus y \otimes tk[t] \oplus z \otimes (t-1)k[t].$$

3. THE NORMAL LIE RELATED TRIPLE ALGEBRA ASSOCIATED TO THE TETRAHEDRON ALGEBRA

Following [EO05], given the Lie algebra \mathfrak{g} on which S_4 acts as automorphisms, there exists a structure of normal Lie related triple algebra defined on \mathfrak{g}_0 , which essentially determines \mathfrak{g} .

A *normal Lie related triple algebra* $(A, \cdot, \bar{\cdot})$ is an algebra with multiplication \cdot , with an involution $\bar{\cdot}$, and endowed with a skew-symmetric bilinear map $\delta : A \times A \rightarrow \mathfrak{ltt}(A, \cdot, \bar{\cdot})$, where

$$\begin{aligned} \mathfrak{ltt}(A, \cdot, \bar{\cdot}) &= \{(d_0, d_1, d_2) \in \mathfrak{gl}(A)^3 : \bar{d}_i(x \cdot y) = d_{i+1}(x) \cdot y + x \cdot d_{i+2}(y) \\ &\quad \text{for any } x, y \in A \text{ and } i \in \mathbb{Z}_3\}, \end{aligned}$$

satisfying some conditions (see [Oku05, (2.34)] for a complete definition).

Here $\mathfrak{g}_0 = u_0 \mathcal{A}$ can be identified with \mathcal{A} by means of $\iota_0 : \mathcal{A} \rightarrow \mathfrak{g}_0$, $a \mapsto \iota_0(a) = u_0 a$. Then, according to [EO05], one has to consider the identifications $\iota_i : \mathcal{A} \rightarrow \mathfrak{g}_i$ given by

$$\iota_i(a) = \varphi^i(\iota_0(a)) = \varphi^i(u_0 a) = \varphi^i(u_0) \varphi_{\mathcal{A}}^i(a) = u_i \varphi_{\mathcal{A}}^i(a),$$

for $i = 0, 1, 2$ (see Proposition 1.4). Therefore, by Theorem 2.2,

$$\mathfrak{g} = \oplus_{i=0}^2 \iota_i(\mathcal{A}),$$

and the action of \mathcal{A} on \mathfrak{g} is given by

$$\iota_i(a)b = u_i\varphi_{\mathcal{A}}^i(a)b = u_i\varphi_{\mathcal{A}}^i(a\varphi_{\mathcal{A}}^{-i}(b)) = \iota_i(a\varphi_{\mathcal{A}}^{-i}(b)),$$

for any $a, b \in \mathcal{A}$ and $i = 0, 1, 2$.

Now, according to [EO05, Section 2], the k -vector space \mathcal{A} is endowed with an involution $a \mapsto \bar{a}$ determined by $\iota_0(\bar{a}) = -\tau(\iota_0(a))$. That is, by Proposition 1.4,

$$\begin{aligned} u_0\bar{a} &= -\tau(u_0a) = -\tau(u_0)\tau_{\mathcal{A}}(a) \\ &= -\frac{1}{4}\tau(z \otimes 1 + x \otimes t'' + y \otimes (t'' - 1))\tau_{\mathcal{A}}(a) \\ &= -\frac{1}{4}(\tau_{\mathfrak{s}}(z) \otimes 1 + \tau_{\mathfrak{s}}(x) \otimes \tau_{\mathcal{A}}(t'') + \tau_{\mathfrak{s}}(y) \otimes \tau_{\mathcal{A}}(t'' - 1))\tau_{\mathcal{A}}(a) \\ &= -\frac{1}{4}(-y \otimes 1 - x \otimes (1 - t') + z \otimes t')\tau_{\mathcal{A}}(a) \\ &= -u_0t'\tau_{\mathcal{A}}(a), \end{aligned}$$

as $t''t' = t' - 1$. Therefore,

$$\bar{a} = -t'\tau_{\mathcal{A}}(a)$$

for any $a \in \mathcal{A}$. Note that since $\tau^2 = 1$, $\bar{\bar{a}} = a$ for any $a \in \mathcal{A}$.

Also, \mathcal{A} is endowed with a multiplication determined by

$$\iota_0(\overline{a \cdot b}) = [\iota_1(a), \iota_2(b)].$$

Hence,

$$\begin{aligned} u_0(\overline{a \cdot b}) &= \iota_0(\overline{a \cdot b}) = [\iota_1(a), \iota_2(b)] \\ &= [u_1\varphi_{\mathcal{A}}(a), u_2\varphi_{\mathcal{A}}^2(b)] \\ &= [u_1, u_2]\varphi_{\mathcal{A}}(a)\varphi_{\mathcal{A}}^2(b) \\ &= -u_0t'\varphi_{\mathcal{A}}(a)\varphi_{\mathcal{A}}^2(b) \end{aligned}$$

(see Theorem 1.9), so

$$\begin{aligned} a \cdot b &= \overline{-t'\varphi_{\mathcal{A}}(a)\varphi_{\mathcal{A}}^2(b)} \\ &= t'\tau_{\mathcal{A}}(t'\varphi_{\mathcal{A}}(a)\varphi_{\mathcal{A}}^2(b)) \\ &= (\tau_{\mathcal{A}}\varphi_{\mathcal{A}}(a))(\tau_{\mathcal{A}}\varphi_{\mathcal{A}}^2(b)), \end{aligned}$$

for any $a, b \in \mathcal{A}$ (since $\tau_{\mathcal{A}}(t') = (t')^{-1}$).

Summarizing, we have obtained:

Proposition 3.1. *The normal Lie related triple algebra associated to the S_4 -action on the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$ is isomorphic to $(\mathcal{A}, \cdot, \bar{\cdot})$, where*

$$\begin{cases} a \cdot b = (\tau_{\mathcal{A}}\varphi_{\mathcal{A}}(a))(\tau_{\mathcal{A}}\varphi_{\mathcal{A}}^2(b)) \\ \bar{a} = -t'\tau_{\mathcal{A}}(a) \end{cases}$$

for any $a, b \in \mathcal{A}$.

This result highlights a family of normal Lie related triple algebras, whose associated Lie algebras with S_4 -action satisfy that there are no nonzero elements fixed by Klein's 4 group:

Proposition 3.2. *Let A be a unital commutative associative algebra endowed with a group homomorphism $\theta : S_3 \rightarrow \text{Aut}(A)$ (the image of any σ under θ will be denoted by σ too), and an element $s \in A$ such that $s\varphi(s)\varphi^2(s) = 1 = s\tau(s)$ (notation as in Section 1). Define on A a new multiplication by*

$$a \cdot b = (\tau\varphi(a))(\tau\varphi^2(b))$$

for any $a, b \in A$, and a linear map $A \rightarrow A$, $a \mapsto \bar{a}$, by

$$\bar{a} = s\tau(a).$$

Then $(A, \cdot, \bar{\cdot})$ is a normal Lie related triple algebra with trivial associated bilinear map $\delta : A \times A \rightarrow \text{lt}(A, \cdot, \bar{\cdot})$.

Proof. Let us first check that $a \mapsto \bar{a}$ is an involution of (A, \cdot) . For any $a \in A$,

$$\bar{\bar{a}} = s\tau(\bar{a}) = s\tau(s\tau(a)) = s\tau(s)\tau^2(a) = a,$$

while for any $a, b \in A$,

$$\begin{aligned} \bar{b} \cdot \bar{a} &= (\tau\varphi(\bar{b}))(\tau\varphi^2(\bar{a})) \\ &= (\tau\varphi(s\tau(b)))(\tau\varphi^2(s\tau(a))) \\ &= \tau(\varphi(s)\varphi^2(s))\varphi^2(b)\varphi(a) \quad (\text{as } \tau\varphi\tau = \varphi^2) \\ &= s\tau(s\varphi(s)\varphi^2(s))\varphi(a)\varphi^2(b) \quad (\text{as } s\tau(s) = 1) \\ &= s\varphi(a)\varphi^2(b) \quad (\text{as } s\varphi(s)\varphi^2(s) = 1) \\ &= s\tau((\tau\varphi(a))(\tau\varphi^2(b))) \\ &= \overline{a \cdot b}. \end{aligned}$$

Now, because of [EO05, Theorem 2.4], for any $a, b, c \in A$,

$$\begin{aligned} \bar{b} \cdot (a \cdot c) &= (s\tau(b)) \cdot ((\tau\varphi(a))(\tau\varphi^2(b))) \\ &= (\tau\varphi(s\tau(b)))(\tau\varphi^2(\tau\varphi(a)\varphi^2(c))) \\ &= (\tau\varphi(s))\varphi^2(b)\varphi(\varphi(a)\varphi^2(c)) \\ &= (\tau\varphi(s))\varphi^2(ab)c, \end{aligned}$$

which is symmetric on a and b . Therefore,

$$\delta_1(a, b)(c) = \bar{b} \cdot (a \cdot c) - \bar{a} \cdot (b \cdot c) = 0$$

and, similarly,

$$\delta_2(a, b)(c) = (c \cdot a) \cdot \bar{b} - (c \cdot b) \cdot \bar{a} = 0,$$

for any $a, b, c \in A$. This shows that $(A, \cdot, \bar{\cdot})$ is trivially ($\delta = 0$) a normal Lie related triple algebra. \square

In our case, with $\mathcal{A} = k[t, t^{-1}, (1-t)^{-1}]$, the action of S_3 is determined by the automorphisms $\varphi_{\mathcal{A}}$ and $\tau_{\mathcal{A}}$ in Proposition 1.4, and the distinguished element s is $-t'$, which satisfies $s\varphi_{\mathcal{A}}(s)\varphi_{\mathcal{A}}^2(s) = -t't''t = 1 = s\tau_{\mathcal{A}}(s)$.

4. THE SOLUTION TO [HT05, Problem 13.3]

In this section, the ideals of both the Tetrahedron algebra and the Onsager algebra will be determined. Let us first deal with the Tetrahedron algebra, whose determination constitutes the problem posed in [HT05, Problem 13.3].

Proposition 4.1. \mathfrak{g}_{\boxtimes} is a prime Lie algebra.

Proof. Since $\mathfrak{sl}_2 \otimes \mathcal{A} \otimes_{\mathcal{A}} k(t)$ is isomorphic to the simple Lie algebra $\mathfrak{sl}_2 \otimes k(t)$, it follows that $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$ is prime, and hence so is \mathfrak{g}_{\boxtimes} . \square

Let \mathfrak{a} be an ideal of $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$. The aim is to prove that there is an ideal \mathcal{I} of \mathcal{A} such that

$$\mathfrak{a} = u_0\mathcal{I} \oplus u_1\mathcal{I} \oplus u_2\mathcal{I}.$$

Therefore, the set of ideals of \mathfrak{g} (and hence of \mathfrak{g}_{\boxtimes}) is in bijection with the set of ideals of \mathcal{A} .

To prove this, take any element $u_0a + u_1b + u_2c$ in \mathfrak{g} ($a, b, c \in \mathcal{A}$). Then, using Theorem 1.9 and since $tt't'' = -1$:

$$\begin{aligned} [u_0, [u_1, [u_2, u_0a + u_1b + u_2c]]] &= -u_1b, \\ [u_1, [u_2, [u_0, u_0a + u_1b + u_2c]]] &= -u_2c, \\ [u_2, [u_0, [u_1, u_0a + u_1b + u_2c]]] &= -u_0a. \end{aligned} \tag{4.2}$$

Theorem 4.3. The ideals of $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$ are precisely the subspaces

$$u_0\mathcal{I} \oplus u_1\mathcal{I} \oplus u_2\mathcal{I},$$

where $\{u_0, u_1, u_2\}$ is the \mathcal{A} -basis of \mathfrak{g} in Theorem 1.9 and \mathcal{I} is an ideal of \mathcal{A} .

Proof. Let \mathfrak{a} be an ideal of \mathfrak{g} and consider the subspaces $\mathcal{I}_i = \{a \in \mathcal{A} : u_i a \in \mathfrak{a}\}$, $i = 0, 1, 2$. As in (4.2), $[u_2b, [u_0, [u_1, u_0a]]] = -u_0ba$, so \mathcal{I}_0 is an ideal of \mathcal{A} , and so are \mathcal{I}_1 and \mathcal{I}_2 . Now, because of (4.2), $\mathfrak{a} = u_0\mathcal{I}_0 \oplus u_1\mathcal{I}_1 \oplus u_2\mathcal{I}_2$. But for any $a \in \mathcal{I}_0$ and $b \in \mathcal{A}$, $[u_2b, u_0a] = -u_1bt''a \in \mathfrak{a}$, so $\mathcal{A}t''a \subseteq \mathcal{I}_1$. Since t'' is invertible in \mathcal{A} , this shows that $\mathcal{I}_0 = \mathcal{A}\mathcal{I}_0 = \mathcal{A}t''\mathcal{I}_0 \subseteq \mathcal{I}_1$. In the same vein, one proves $\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_0$, so $\mathcal{I}_0 = \mathcal{I}_1 = \mathcal{I}_2$ and the result follows. \square

Notice that \mathcal{A} , as a ring of fractions of $k[t]$, is a principal ideal domain.

Actually, a much more general result can be given:

Theorem 4.4. Let \mathfrak{g} be a central simple finite dimensional Lie algebra, and let A be a unital commutative associative algebra. Then the ideals of $\mathfrak{g} \otimes A$ are precisely the subspaces $\mathfrak{g} \otimes I$, where I is an ideal of A ($\mathfrak{g} \otimes I$ is naturally identified with a subspace of $\mathfrak{g} \otimes A$).

Proof. Let $\mathcal{M}(\mathfrak{g})$ be the associative subring (not necessarily unital) of $\text{End}_k(\mathfrak{g})$ generated by $\{\text{ad}_g : g \in \mathfrak{g}\}$ (see [Jac79, Chapter X]). A well-known result by Wedderburn shows that $\mathcal{M}(\mathfrak{g}) = \text{End}_k(\mathfrak{g})$, since \mathfrak{g} , as a central simple Lie algebra, is an irreducible module for $\mathcal{M}(\mathfrak{g})$, and the centralizer of the action is k . Let $\{g_i : i = 1, \dots, n\}$ be a basis of \mathfrak{g} and let $e_{ij} \in \text{End}_k(\mathfrak{g})$ be the linear map given by $e_{ij}(g_k) = \delta_{jk}g_i$. Then, for any $x = \sum_{i=1}^n g_i \otimes a_i \in \mathfrak{g} \otimes A$ ($a_i \in A$, $i = 1, \dots, n$), $g_i \otimes a_i = (e_{ii} \otimes 1)(x)$ and $g_j \otimes a_i b = (e_{ji} \otimes b)(x)$ belong to the ideal of $\mathfrak{g} \otimes A$ generated by x , as both e_{ii} and e_{ji} belong to $\mathcal{M}(\mathfrak{g})$. Now, if \mathfrak{a} is an ideal of $\mathfrak{g} \otimes A$, and $I = \{a \in A : g_1 \otimes a \in \mathfrak{a}\}$, the

above arguments show that, for any $i = 1, \dots, n$, $I = \{a \in A : g_i \otimes a \in \mathfrak{a}\}$, I is an ideal of A , and $\mathfrak{a} = \mathfrak{g} \otimes I$. \square

In [DR00], an ideal \mathfrak{a} of a Lie algebra \mathfrak{g} is said to be *closed* if $Z(\mathfrak{a}) = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subseteq \mathfrak{a}\}$ equals \mathfrak{a} . Note that $Z(\mathfrak{a})/\mathfrak{a}$ is the center of $\mathfrak{g}/\mathfrak{a}$, so the ideal \mathfrak{a} is closed if the center of $\mathfrak{g}/\mathfrak{a}$ is trivial.

Corollary 4.5. *Let \mathfrak{g} be a central simple finite dimensional Lie algebra, and let A be a unital commutative associative algebra. Then any ideal of $\mathfrak{g} \otimes A$ is closed.*

Proof. From Theorem 4.4 it follows that any ideal of $\mathfrak{g} \otimes A$ is of the form $\mathfrak{g} \otimes I$ for some ideal I of A , and hence the quotient $\mathfrak{g} \otimes A / \mathfrak{g} \otimes I$ is isomorphic to $\mathfrak{g} \otimes (A/I)$, whose center is trivial, as A/I is unital. \square

The closed ideals of the Onsager algebra have been determined in [DR00] and [DR00']. Here all the ideals of the Onsager algebra will be determined. To do so, let us identify the Onsager algebra \mathcal{O} with $\psi(\Omega) = u_0(t-1)k[t] \oplus u_1k[t] \oplus u_2tk[t]$ (Proposition 1.15), which is closed under the action of Klein's 4 group. Consider the following elements in \mathcal{O} :

$$v_0 = u_0(t-1), \quad v_1 = u_1, \quad v_2 = u_2t,$$

which are free generators of \mathcal{O} over $k[t]$, and satisfy

$$[v_0, v_1] = -v_2(t-1), \quad [v_1, v_2] = -v_0, \quad [v_2, v_0] = v_1t, \quad (4.6)$$

because of Theorem 1.9, as $t't = t-1$ and $t''(t-1) = -1$.

Recall that the centroid of a Lie algebra \mathfrak{g} over k is the centralizer of the adjoint action: $\Gamma(\mathfrak{g}) = \{f \in \text{End}_k(\mathfrak{g}) : f([x, y]) = [x, f(y)] \text{ for any } x, y \in \mathfrak{g}\}$.

Lemma 4.7.

- (i) \mathcal{O} is generated, as an algebra over k , by v_0, v_1 and v_2 .
- (ii) The centroid of \mathcal{O} is isomorphic to $k[t]$.
- (iii) \mathcal{O} is prime.

Proof. Since $(\text{ad}_{v_2})^2(v_0t^n) = v_0t^{n+1}$, it follows that v_0t^n belongs to the subalgebra generated by v_0, v_1 and v_2 for any n . But then so does $v_1t^{n+1} = [v_2, v_0t^n]$ and $v_2t^n(t-1) = [v_1, v_0t^n]$. Hence (i) follows.

It is clear that $k[t]$ embeds in the centroid $\Gamma(\mathcal{O})$, since \mathcal{O} is an algebra over $k[t]$. But for any $f \in \Gamma(\mathcal{O})$, $f(v_i) \in \{x \in \mathcal{O} : [x, v_i] = 0\} = v_i k[t]$, so there are polynomials $p_i(t) \in k[t]$ such that $f(v_i) = v_i p_i(t)$, $i = 0, 1, 2$. Then

$$v_0 p_0(t) = f(v_0) = f([v_2, v_1]) = \begin{cases} [f(v_2), v_1] = [v_2 p_2(t), v_1] = v_0 p_2(t), \\ [v_2, f(v_1)] = [v_2, v_1 p_1(t)] = v_0 p_1(t), \end{cases}$$

so $p_0(t) = p_1(t) = p_2(t)$. Because of (i), f is determined by its action on v_0, v_1 and v_2 , and hence f is the right multiplication by $p_0(t)$.

Finally, $\mathcal{O} \otimes_{k[t]} k(t)$ is a three dimensional simple algebra over the field $k(t)$, so that \mathcal{O} is prime. \square

In order to determine the ideals of \mathcal{O} , first note that for any ideal J of $k[t]$, $\mathcal{O}J$ is an ideal of \mathcal{O} . These ideals are closed under the action of Klein's 4 group.

Let now \mathcal{I} be an ideal of \mathcal{O} , and consider the following subspace of $k[t]$:

$$J_{\mathcal{I}} = \{p(t) \in k[t] : v_0p(t) + v_1p_1(t) + v_2p_2(t) \in \mathcal{I} \text{ for some } p_1(t), p_2(t) \in k[t]\}.$$

Proposition 4.8. *Let \mathcal{I} be an ideal of \mathcal{O} . Then:*

- (i) $J_{\mathcal{I}}$ is an ideal of $k[t]$.
- (ii) \mathcal{I} lies between the ideals $\mathcal{O}J_{\mathcal{I}}t(t-1)$ and $\mathcal{O}J_{\mathcal{I}}$:

$$\mathcal{O}J_{\mathcal{I}}t(t-1) \subseteq \mathcal{I} \subseteq \mathcal{O}J_{\mathcal{I}}.$$

Proof. For any $p(t) \in J = J_{\mathcal{I}}$, there are polynomials $p_1(t), p_2(t) \in k[t]$ such that $x = v_0p(t) + v_1p_1(t) + v_2p_2(t)$ belongs to \mathcal{I} . But

$$(\text{ad}_{v_2})^2(x) = v_0tp(t) + v_1tp_1(t) \in \mathcal{I},$$

so $tp(t) \in J$, and hence J is an ideal of $k[t]$.

Now, if $x = v_0p_0(t) + v_1p_1(t) + v_2p_2(t) \in \mathcal{I}$, then $p_0(t)$ lies in J , but since

$$\begin{aligned} [v_2, x] &= v_0p_1(t) + v_1tp_0(t) \in \mathcal{I}, \\ [-v_1, x] &= v_0p_2(t) - v_2p_0(t)(t-1) \in \mathcal{I}, \end{aligned}$$

it follows that both $p_1(t)$ and $p_2(t)$ lie in J too. Therefore, \mathcal{I} is contained in $v_0J \oplus v_1J \oplus v_2J = \mathcal{O}J$. Moreover,

$$\begin{aligned} [v_0, [v_1, x]] &= v_1p_0(t)t(t-1) \in \mathcal{I}, \\ [v_2, [v_0, [v_1, x]]] &= v_0p_0(t)t(t-1) \in \mathcal{I}, \\ [-v_0, [v_2, x]] &= v_2p_0(t)t(t-1) \in \mathcal{I}, \end{aligned}$$

so $\mathcal{O}Jt(t-1) = v_0Jt(t-1) \oplus v_1Jt(t-1) \oplus v_2Jt(t-1)$ is contained in \mathcal{I} . \square

Corollary 4.9. *Let \mathcal{I} be a maximal ideal of \mathcal{O} , then the quotient algebra \mathcal{O}/\mathcal{I} is either a one dimensional Lie algebra over k , or a three dimensional simple Lie algebra over a finite field extension of k .*

Proof. By maximality, either $J_{\mathcal{I}} = k[t]$, or $\mathcal{I} = \mathcal{O}J_{\mathcal{I}}$ and $J = J_{\mathcal{I}}$ is a maximal ideal of $k[t]$. In the first case, $v_0 \in \mathcal{I}$, and hence $v_2(t-1) = [v_1, v_0] \in \mathcal{I}$ and $v_1t = [v_2, v_0] \in \mathcal{I}$. Thus \mathcal{O}/\mathcal{I} is spanned by $\bar{v}_1 = v_1 + \mathcal{I}$ and $\bar{v}_2 = v_2 + \mathcal{I}$, which satisfy $[\bar{v}_1, \bar{v}_2] = 0$. It follows that \mathcal{O}/\mathcal{I} is abelian and, by maximality of \mathcal{I} , the dimension of \mathcal{O}/\mathcal{I} is 1.

Otherwise $\mathcal{I} = \mathcal{O}J_{\mathcal{I}}$ and $J = J_{\mathcal{I}}$ is a maximal ideal of $k[t]$. Then there are three different possibilities, according to J being the ideal generated by t , by $t-1$, or by a monic irreducible polynomial different from t or $t-1$.

If $J = tk[t]$, then \mathcal{O}/\mathcal{I} is spanned by $\bar{v}_i = v_i + \mathcal{I}$, $i = 0, 1, 2$, which satisfy $[\bar{v}_2, \bar{v}_0] = 0$, $[\bar{v}_1, \bar{v}_2] = -\bar{v}_0$ and $[\bar{v}_0, \bar{v}_1] = \bar{v}_2$, thus giving a three dimensional solvable Lie algebra, which contradicts the maximality of \mathcal{I} . The same happens if $J = (t-1)k[t]$.

However, if $J = p(t)k[t]$, for a monic irreducible polynomial different from t and $t-1$, then $K = k[t]/J$ is a finite field extension of k , and \mathcal{O}/\mathcal{I} is naturally a Lie algebra over K with a basis $\{\bar{v}_i = v_i + \mathcal{I} : i = 0, 1, 2\}$. Because of (4.6), this is a simple three dimensional Lie algebra over K . \square

Note that in $\mathcal{A} = k[t, t^{-1}, (1-t)^{-1}]$, both t and $t-1$ are invertible, and hence, with the same arguments as in the previous proof, it is easily checked that the quotient of the Tetrahedron algebra \mathfrak{g}_{\boxtimes} by any maximal ideal is always a three dimensional simple Lie algebra over a finite field extension of k .

Note that $k[t]$ decomposes as

$$k[t] = t(t-1)k[t] \oplus (kt \oplus k(t-1))$$

(direct sum of subspaces). Therefore,

$$\mathcal{O} = \mathcal{O}t(t-1) \oplus \text{span} \{v_i t, v_i(t-1) : i = 0, 1, 2\}, \quad (4.10)$$

and if $J = q(t)k[t]$ is a nonzero ideal of $k[t]$, then

$$\mathcal{O}J = \mathcal{O}Jt(t-1) \oplus \text{span} \{v_i q(t)t, v_i q(t)t(t-1) : i = 0, 1, 2\}.$$

Besides, for any nonzero ideal J of $k[t]$, $[\mathcal{O}t(t-1), \mathcal{O}J]$ is contained in $\mathcal{O}Jt(t-1)$, so there are natural bijections

$$\begin{aligned} & \{\text{ideals } \mathcal{I} \text{ of } \mathcal{O} \text{ with } \mathcal{O}Jt(t-1) \subseteq \mathcal{I} \subseteq \mathcal{O}J\} \\ & \quad \updownarrow \\ & \{\mathcal{O}\text{-submodules of } \mathcal{O}J/\mathcal{O}Jt(t-1)\} \\ & \quad \updownarrow \\ & \{\mathcal{O}/\mathcal{O}t(t-1)\text{-submodules of } \mathcal{O}J/\mathcal{O}Jt(t-1)\} \end{aligned}$$

Given an element $x \in \mathcal{O}$ (respectively $x \in \mathcal{O}J$), let us denote by \bar{x} its class modulo $\mathcal{O}t(t-1)$ (respectively, modulo $\mathcal{O}Jt(t-1)$). Thus, from (4.10), with $J = q(t)k[t] \neq 0$ and $w_i = v_i q(t)$, $i = 0, 1, 2$,

$$\begin{aligned} \mathcal{O}/\mathcal{O}t(t-1) &= \bigoplus_{i=0}^2 (\overline{kv_i t} \oplus \overline{kv_i(t-1)}), \\ \mathcal{O}J/\mathcal{O}Jt(t-1) &= \bigoplus_{i=0}^2 (\overline{kw_i t} \oplus \overline{kw_i(t-1)}). \end{aligned}$$

Note also that for any i, j , $[\overline{v_i t}, \overline{w_j(t-1)}] = 0 = [\overline{v_i(t-1)}, \overline{w_j t}]$, since $[v_i t, w_j(t-1)], [v_i(t-1), w_j t] \in \mathcal{O}Jt(t-1)$.

The eigenvalues of the action of $\overline{v_2 t}$ on $\mathcal{O}J/\mathcal{O}Jt(t-1)$ are:

- 0, with eigenspace $\overline{kw_2 t} \oplus (\oplus_{i=0}^2 \overline{kw_i(t-1)})$,
- 1, with eigenspace $\overline{kw_0 t + w_1 t}$, and
- -1, with eigenspace $\overline{kw_0 t - w_1 t}$.

(For instance, by (4.6), $[v_2 t, w_0 t + w_1 t] = w_1 t^3 + w_0 t^3$, but $t^3 = t(t-1)(t+1) + t$, so $\overline{w_i t^3} = \overline{w_i t}$.)

Also, the eigenvalues of the action of $\overline{v_1(t-1)}$ on $\mathcal{O}J/\mathcal{O}Jt(t-1)$ are:

- 0, with eigenspace $(\oplus_{i=0}^2 \overline{kw_i t}) \oplus \overline{kw_1(t-1)}$,
- 1, with eigenspace $\overline{kw_0(t-1) + w_2(t-1)}$, and
- -1, with eigenspace $\overline{kw_0(t-1) - w_2(t-1)}$.

Besides, $[v_0t, w_1t] = -w_2t^2(t-1)$ belongs to $\mathcal{O}Jt(t-1)$, and the same happens to $[v_1t, w_0t]$, $[v_0(t-1), w_2(t-1)]$ and $[v_2(t-1), w_0(t-1)]$. Moreover, $\overline{w_2t}$ generates the \mathcal{O} -submodule $\oplus_{i=0}^2 k\overline{w_it}$, and $\overline{w_1(t-1)}$ generates $\oplus_{i=0}^2 k\overline{w_i(t-1)}$.

Therefore, since any \mathcal{O} -submodule of $\mathcal{O}J/\mathcal{O}Jt(t-1)$ is the direct sum of its intersections with the previous eigenspaces, we get:

Proposition 4.11. *Let $J = q(t)k[t]$ be a nonzero ideal of $k[t]$. Then the ideals \mathcal{I} of \mathcal{O} with $J = J_{\mathcal{I}}$ are the subspaces*

$$\mathcal{I} = \mathcal{O}Jt(t-1) \oplus \mathcal{S},$$

where \mathcal{S} is of one of the following types:

- (i) $\mathcal{S} = k\epsilon(w_0t + w_1t) \oplus k\delta(w_0t - w_1t) \oplus k\epsilon\delta\gamma w_2t \oplus k\epsilon'(w_0(t-1) + w_2(t-1)) \oplus k\delta'(w_0(t-1) - w_2(t-1)) \oplus k\epsilon'\delta'\gamma'w_1(t-1)$, where $w_i = v_iq(t)$, $i = 0, 1, 2$, and $\epsilon, \delta, \gamma, \epsilon', \delta', \gamma'$ are either 0 or 1, with $\epsilon + \delta \neq 0 \neq \epsilon' + \delta'$ (as $J = J_{\mathcal{I}}$).
- (ii) $\mathcal{S} = \mathcal{S}_{\eta} = \text{span}\{w_0t, w_1t, w_0(t-1), w_2(t-1), w_2t + \eta w_1(t-1)\}$, with $0 \neq \eta \in k$.

Remark 4.12. The ideals in Proposition 4.11 which are invariant under the action of Klein's 4 group are those of type (i) with $\epsilon = \delta = \epsilon' = \delta' = 1$. In this case, $\mathcal{I} = v_0J \oplus v_1J_1 \oplus v_2J_2$, where J_1 is either J or Jt , and J_2 is either J or $J(t-1)$ (four possibilities which correspond to γ and γ' being either 0 or 1).

Recall that an ideal \mathcal{I} of \mathcal{O} is closed if $Z(\mathcal{I}) = \{x \in \mathcal{O} : [x, \mathcal{O}] \subseteq \mathcal{I}\}$ equals \mathcal{I} . Write $J = J_{\mathcal{I}}$. Then, for any $x = v_0p_0(t) + v_1p_1(t) + v_2p_2(t)$ in $Z(\mathcal{I})$,

$$[x, v_1] = -v_2(t-1)p_0(t) + v_0p_2(t) \in \mathcal{I} \subseteq \mathcal{O}J,$$

$$[x, v_2] = -v_1tp_0(t) - v_0p_1(t) \in \mathcal{I} \subseteq \mathcal{O}J,$$

and hence $p_1(t), p_2(t) \in J$ and $(t-1)p_0(t), tp_0(t) \in J$, so $p_0(t) = tp_0(t) - (t-1)p_0(t) \in J$ too. Therefore,

$$\mathcal{O}Jt(t-1) \subseteq \mathcal{I} \subseteq Z(\mathcal{I}) \subseteq \mathcal{O}J.$$

Also, if $J = q(t)k[t]$ and $w_i = v_iq(t)$, $i = 0, 1, 2$, using (4.10), we get

$$\begin{aligned} [w_2t, \mathcal{O}] &= [w_2t, \mathcal{O}t(t-1)] + k[w_2t, v_0t] + k[w_2t, v_1t] \\ &\quad + \sum_{i=0}^2 k[w_2t, v_i(t-1)] \\ &\subseteq \mathcal{O}Jt(t-1) + kw_1t + kw_0t, \end{aligned}$$

and this is contained in \mathcal{I} in case \mathcal{I} is as in item (ii) of Proposition 4.11, so $w_2t \in Z(\mathcal{I}) \setminus \mathcal{I}$ in this case. The same happens for \mathcal{I} as in item (i) of Proposition 4.11 with $\epsilon = \delta = 1$ and $\gamma = 0$, or with $\epsilon' = \delta' = 1$ and $\gamma' = 0$. For the remaining ideals in Proposition 4.11, it is easily checked that they are closed. Therefore:

Proposition 4.13. *Let \mathcal{I} be a closed ideal of \mathcal{O} , with $0 \neq J = J_{\mathcal{I}} = q(t)k[t]$, then, with $w_i = v_ik[t]$, $i = 0, 1, 2$, \mathcal{I} is one of the following ideals:*

- (a) $\mathcal{I} = \mathcal{O}Jt(t-1) \oplus k(w_0t \pm w_1t) \oplus k(w_0(t-1) \pm w_2(t-1))$ (4 possibilities).

- (b) $\mathcal{I} = \mathcal{O}Jt(t-1) \oplus (\oplus_{i=0}^2 kw_it) \oplus k(w_0(t-1) \pm w_2(t-1))$ (2 possibilities).
In this case $\mathcal{I} = \mathcal{O}Jt \oplus k(w_0 \pm w_2)$.
- (c) $\mathcal{I} = \mathcal{O}Jt(t-1) \oplus k(w_0t \pm w_1t) \oplus (\oplus_{i=0}^2 kw_i(t-1))$ (2 possibilities).
In this case $\mathcal{I} = \mathcal{O}J(t-1) \oplus k(w_0 \pm w_1)$.
- (d) $\mathcal{I} = \mathcal{O}J$.

Proof. Only the last assertions in (b) and (c) need to be checked. Since $Jt = Jt(t-1) \oplus kq(t)t$, it follows that $\mathcal{O}Jt = \mathcal{O}Jt(t-1) \oplus (\oplus_{i=0}^2 kw_it)$. Besides, $w_i(t-1) + w_i = w_it \in \mathcal{O}Jt$. The last assertion in item (b) follows at once. The argument for item (c) is similar. \square

Note that only the ideal in (d) is invariant under the action of Klein's 4 group.

5. ANOTHER PRESENTATION OF THE TETRAHEDRON ALGEBRA

In this section, a new presentation of the Tetrahedron algebra, based on the properties of our basis over \mathcal{A} , will be given.

Consider the Lie algebra \mathfrak{f} generated by the elements z_0, z_1 and z_2 subject to the relations:

$$[[z_i, z_{i+1}], z_{i+2}] = 0, \quad (5.1a)$$

$$[z_i, [z_i, z_{i+1}]] = z_{i+1} + [z_{i+2}, z_i], \quad (5.1b)$$

$$[[z_{i+1}, [z_{i+1}, [z_{i+1}, z_i]]], [z_{i+1}, z_i]] = 0, \quad (5.1c)$$

for any $i \in \{0, 1, 2\}$, and where the indices are taken modulo 3.

The aim of this section is to show that \mathfrak{f} is isomorphic to the Tetrahedron algebra \mathfrak{g}_{\boxtimes} , thus providing a different presentation of this latter algebra.

First, let us obtain some consequences of the relations (5.1):

Lemma 5.2. *For any $i, j \in \{0, 1, 2\}$, and taking indices modulo 3, the following relations are satisfied:*

$$[(\text{ad}_{z_i})^2(z_j), z_j] = 0, \quad (5.3)$$

$$[[z_{i-1}, z_i], [z_i, z_{i+1}]] = z_i, \quad (5.4)$$

$$(\text{ad}_{[z_i, z_{i+1}]}^3)(z_{i+1}) = (\text{ad}_{z_{i+1}})^2(z_i) - (\text{ad}_{z_{i+1}})^4(z_i). \quad (5.5)$$

Proof. Equation (5.3) is clear for $i = j$. Also,

$$[(\text{ad}_{z_i})^2(z_{i+1}), z_{i+1}] = [z_{i+1} + [z_{i+2}, z_i], z_{i+1}] = 0,$$

because of (5.1b) and (5.1a), while

$$[(\text{ad}_{z_i})^2(z_{i-1}), z_{i-1}] = [z_i, [[z_i, z_{i-1}], z_{i-1}]] = -[(\text{ad}_{z_{i-1}})^2(z_i), z_i] = 0$$

by the previous argument. Hence (5.3) holds.

Now,

$$\begin{aligned} [[z_{i-1}, z_i], [z_i, z_{i+1}]] &= [z_{i-1}, [z_i, [z_i, z_{i+1}]]] && \text{(because of (5.1a))} \\ &= [z_{i-1}, z_{i+1} + [z_{i+2}, z_i]] && \text{(using (5.1b))} \\ &= [z_{i-1}, z_{i+1}] + [z_{i-1}, [z_{i+2}, z_i]] \\ &= [z_{i-1}, z_{i+1}] + z_i + [z_{i+1}, z_{i-1}] && \text{(using again (5.1b))} \\ &= z_i. \end{aligned}$$

thus proving (5.4).

Finally,

$$\begin{aligned}
& (\text{ad}_{[z_i, z_{i+1}]})^3(z_{i+1}) \\
&= \left[[z_i, z_{i+1}], [[z_i, z_{i+1}], [[z_i, z_{i+1}], z_{i+1}]] \right] \\
&= - \left[[z_i, z_{i+1}], [z_i, [[[z_i, z_{i+1}], z_{i+1}], z_{i+1}]] \right] \quad (\text{by (5.3)}) \\
&= \left[[z_i, [z_i, z_{i+1}]], [[[z_i, z_{i+1}], z_{i+1}], z_{i+1}] \right] \quad (\text{by (5.1c)}) \\
&= - \left[z_{i+1} + [z_{i+2}, z_i], [z_{i+1}, [z_{i+1}, [z_{i+1}, z_i]]] \right] \quad (\text{by (5.1b)}) \\
&= -(\text{ad}_{z_{i+1}})^4(z_i) + \left[z_{i+1}, [z_{i+1}, [[z_{i-1}, z_i], [z_i, z_{i+1}]]] \right] \quad (\text{by (5.1a)}) \\
&= -(\text{ad}_{z_{i+1}})^4(z_i) + (\text{ad}_{z_{i+1}})^2(z_i), \quad (\text{by (5.4)})
\end{aligned}$$

thus proving (5.5). \square

Theorem 5.6. *There is a Lie algebra isomorphism $\Phi : \mathfrak{g}_{\boxtimes} \rightarrow \mathfrak{f}$ such that*

$$\begin{aligned}
\Phi(X_{01}) &= 2(z_2 - [z_1, z_2]), & \Phi(X_{23}) &= 2(z_2 + [z_1, z_2]), \\
\Phi(X_{02}) &= 2(z_0 - [z_2, z_0]), & \Phi(X_{31}) &= 2(z_0 + [z_2, z_0]), \\
\Phi(X_{03}) &= 2(z_1 - [z_0, z_1]), & \Phi(X_{12}) &= 2(z_1 + [z_0, z_1]).
\end{aligned} \tag{5.7}$$

Proof. To check that the formulas above define a Lie algebra homomorphism, it is enough to check the relations (1.2). This is straightforward. For instance, let us check that $[\Phi(X_{01}), \Phi(X_{13})] = 2(\Phi(X_{01}) + \Phi(X_{13}))$ and that $[\Phi(X_{01}), [\Phi(X_{01}), [\Phi(X_{01}), \Phi(X_{23})]]] = 4[\Phi(X_{01}), \Phi(X_{23})]$. First,

$$2(\Phi(X_{01}) + \Phi(X_{13})) = 2(\Phi(X_{01}) - \Phi(X_{31})) = -4([z_1, z_2] + [z_2, z_0]),$$

while

$$\begin{aligned}
& [\Phi(X_{01}), \Phi(X_{13})] \\
&= -4[z_2 - [z_1, z_2], z_0 + [z_2, z_0]] \\
&= -4([z_2, z_0] + [z_2, [z_2, z_0]] - [[z_1, z_2], [z_2, z_1]]) \quad (\text{using (5.1a)}) \\
&= -4([z_2, z_0] + z_2 + [z_1, z_2] - z_2) \quad (\text{because of (5.1b) and (5.4)}) \\
&= -4([z_2, z_0] + [z_1, z_2]).
\end{aligned}$$

Now,

$$\begin{aligned}
4[\Phi(X_{01}), \Phi(X_{23})] &= 16[z_2 - [z_1, z_2], z_0 + [z_2, z_0]] \\
&= -32[z_2, [z_2, z_1]] = -32(\text{ad}_{z_2})^2(z_1),
\end{aligned}$$

while

$$\begin{aligned}
& [\Phi(X_{01}), [\Phi(X_{01}), [\Phi(X_{01}), \Phi(X_{23})]]] \\
&= 16[z_2 - [z_1, z_2], [z_2 - [z_1, z_2], [z_2 - [z_1, z_2], z_2 + [z_1, z_2]]] \\
&= -32[z_2 - [z_1, z_2], [z_2 - [z_1, z_2], [z_2, [z_2, z_1]]] \\
&= -32\left((\text{ad}_{z_2})^4(z_1) + (\text{ad}_{[z_1, z_2]})^3(z_2 \right. \\
&\quad \left. - [[z_1, z_2], [z_2, [z_2, [z_2, z_1]]]] - [z_2, [[z_1, z_2], [z_2, [z_2, z_1]]]]\right) \\
&= -32\left((\text{ad}_{z_2})^2(z_1) - 2[[z_1, z_2], [z_2, [z_2, [z_2, z_1]]]]\right) \quad (\text{by (5.5)}) \\
&= -32(\text{ad}_{z_2})^2(z_1) \quad (\text{because of (5.1c)}).
\end{aligned}$$

Hence (5.7) defines a Lie algebra homomorphism Φ , which is onto, since the generators z_0 , z_1 and z_2 of \mathfrak{f} lie in the image of Φ .

On the other hand, in $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathcal{A}$, Theorem 1.9 shows that,

$$[u_0, [u_0, u_1]] = -[u_0, tu_2] = -tt''u_1 = (1 - t'')u_1 = u_1 + [u_2, u_0],$$

and cyclically. Therefore, the relation (5.1b) is satisfied by the u_i 's, and so are the relations (5.1a) and (5.1c) because of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = u_0\mathcal{A} \oplus u_1\mathcal{A} \oplus u_2\mathcal{A}$ (see (2.1) and Theorem 2.2), where $[\mathfrak{g}_i, \mathfrak{g}_{i+1}] = \mathfrak{g}_{i+2}$ and $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$ for any i .

Hence, there is a surjective homomorphism $\Gamma : \mathfrak{f} \rightarrow \mathfrak{g}$ such that $\Gamma(z_i) = u_i$, $i = 0, 1, 2$. But the composition $\Gamma\Phi$ satisfies

$$\begin{aligned}
& \Gamma\Phi(X_{01} + X_{23}) = 2(u_2 - [u_1, u_2]) + 2(u_2 + [u_1, u_2]) = 4u_2 = \Psi(X_{01} + X_{23}) \\
& (\text{see (1.3) and (1.8)}), \text{ and also } \Gamma\Phi(X_{02} + X_{31}) = 4u_0 = \Psi(X_{02} + X_{31}) \text{ and} \\
& \Gamma\Phi(X_{03} + X_{12}) = 4u_1 = \Psi(X_{03} + X_{12}). \text{ Therefore, } \Gamma\Phi \text{ coincides with } \Psi \\
& \text{on a set of generators, and hence } \Gamma\Phi \text{ coincides with the isomorphism } \Psi. \text{ In} \\
& \text{particular, } \Phi \text{ is one-to-one too.} \quad \square
\end{aligned}$$

Corollary 5.8. *The Tetrahedron algebra \mathfrak{g}_{\boxtimes} is isomorphic to the Lie algebra generated by three elements z_0 , z_1 and z_2 , subject to the relations (5.1).*

6. S_4 -ACTION ON THE UNIVERSAL CENTRAL EXTENSION OF THE TETRAHEDRON ALGEBRA

In [BT, Definition 3.3], a Lie algebra \mathfrak{g}_{\boxtimes} is defined with generators

$$\{\tilde{X}_{ij} : i, j \in \{0, 1, 2, 3\}, i \neq j\} \cup \{C_p : p \in P\},$$

where P is the set of partitions of $\{0, 1, 2, 3\}$ into two disjoint subsets of two elements each, subject to the relations

- (i) C_p is central for any $p \in P$,
- (ii) $\sum_{p \in P} C_p = 0$,
- (iii) $\tilde{X}_{ij} + \tilde{X}_{ji} = C_{ij}$ for $i \neq j$, where $C_{ij} = C_{p_{ij}}$, and p_{ij} consists of $\{i, j\}$ and $\{0, 1, 2, 3\} \setminus \{i, j\}$,

- (iv) $[\tilde{X}_{ij}, \tilde{X}_{jk}] = 2(\tilde{X}_{ij} + \tilde{X}_{jk})$ for mutually distinct i, j, k such that (i, j, k) is even (that is, the permutation σ of $\{0, 1, 2, 3\}$ with $\sigma(0) = i$, $\sigma(1) = j$ and $\sigma(2) = k$ is even).
- (v) $[\tilde{X}_{hi}, [\tilde{X}_{hi}, [\tilde{X}_{hi}, \tilde{X}_{jk}]]] = 4[\tilde{X}_{hi}, \tilde{X}_{jk}]$ for mutually distinct h, i, j, k .

This Lie algebra \mathfrak{g}_{\boxtimes} is a central extension of \mathfrak{g}_{\boxtimes} , and the kernel of the natural projection $\pi : \mathfrak{g}_{\boxtimes} \rightarrow \mathfrak{g}_{\boxtimes}$ ($\tilde{X}_{ij} \mapsto X_{ij}$, $C_p \mapsto 0$) is the two dimensional space spanned by $\{C_p : p \in P\}$. Moreover, if the characteristic of the ground field is 0, then \mathfrak{g}_{\boxtimes} is shown to be the universal central extension of \mathfrak{g}_{\boxtimes} [BT, Theorem 5.3].

The Lie algebra \mathfrak{g}_{\boxtimes} presents a natural A_4 -symmetry, where $\sigma(\tilde{X}_{ij}) = \tilde{X}_{\sigma(i)\sigma(j)}$, for any $i \neq j$, and $\sigma(C_p) = C_{\sigma(p)}$ for any $p \in P$, where $\sigma(p)$ is the partition obtained from p by applying the permutation σ to its two components.

However, the automorphism group of any perfect Lie algebra embeds in the automorphism group of its universal central extension (see [vdK73, Proposition 1.3(v)] or [Pia02, Proposition 2.2]). Therefore, the Lie algebra \mathfrak{g}_{\boxtimes} should show a symmetry over the whole symmetric group S_4 . Let us show how to modify slightly the above generating set of \mathfrak{g}_{\boxtimes} so as to make clear this symmetry.

To do this, consider the new elements

$$Y_{ij} = \tilde{X}_{ij} - \frac{1}{2}C_{ij}$$

for distinct i, j . These elements satisfy

$$Y_{ij} + Y_{ji} = \tilde{X}_{ij} + \tilde{X}_{ji} - C_{ij} = 0$$

by relation (iii) above, and for distinct i, j, k :

$$\begin{aligned} [Y_{ij}, Y_{jk}] &= [\tilde{X}_{ij}, \tilde{X}_{jk}] \\ &= \begin{cases} 2\tilde{X}_{ij} + 2\tilde{X}_{ji} = 2Y_{ij} + 2Y_{ji} + C_{ij} + C_{jk} \\ \quad \quad \quad = 2Y_{ij} + 2Y_{jk} - C_{ik} & \text{for even } (i, j, k) \\ 2\tilde{X}_{ij} + 2\tilde{X}_{ji} + 2C_{ik} = 2Y_{ij} + 2Y_{jk} + C_{ik} & \text{for odd } (i, j, k) \end{cases} \end{aligned}$$

(see [BT, Lemma 3.5]). Hence,

$$[Y_{\sigma(0)\sigma(1)}, Y_{\sigma(1)\sigma(2)}] = 2Y_{\sigma(0)\sigma(1)} + 2Y_{\sigma(1)\sigma(2)} - (-1)^\sigma C_{\sigma(0)\sigma(2)} \quad (6.1)$$

for any $\sigma \in S_4$ ($(-1)^\sigma$ denotes the signature of σ).

Therefore, the generating set $\{Y_{ij} : i, j \in \{0, 1, 2, 3\}, i \neq j\} \cup \{C_p : p \in P\}$ satisfies the relations

- (i') C_p is central for any $p \in P$,
- (ii') $\sum_{p \in P} C_p = 0$,
- (iii') $Y_{ij} + Y_{ji} = 0$ for $i \neq j$,
- (iv') $[Y_{\sigma(0)\sigma(1)}, Y_{\sigma(1)\sigma(2)}] = 2Y_{\sigma(0)\sigma(1)} + 2Y_{\sigma(1)\sigma(2)} - (-1)^\sigma C_{\sigma(0)\sigma(2)}$ for any $\sigma \in S_4$
- (v') $[Y_{hi}, [Y_{hi}, [Y_{hi}, Y_{jk}]]] = 4[Y_{hi}, Y_{jk}]$ for mutually distinct h, i, j, k .

Now, the whole S_4 acts on these generators by $\sigma(Y_{ij}) = Y_{\sigma(i)\sigma(j)}$, for $i \neq j$, and $\sigma(C_p) = (-1)^\sigma C_{\sigma(p)}$ for any $p \in P$. The relations (i')–(v)' above are

invariant under this action of S_4 . Therefore S_4 embeds in the automorphism group $\text{Aut}(\mathfrak{g}_{\boxtimes})$.

Note that for $i \neq j$,

$$\begin{aligned}\sigma(\tilde{X}_{ij}) &= \sigma\left(Y_{ij} + \frac{1}{2}C_{ij}\right) = Y_{\sigma(i)\sigma(j)} + \frac{1}{2}C_{\sigma(i)\sigma(j)} \\ &= \tilde{X}_{\sigma(i)\sigma(j)} - \frac{1}{2}(1 - (-1)^\sigma)C_{\sigma(i)\sigma(j)},\end{aligned}$$

so that

$$\sigma(\tilde{X}_{ij}) = \begin{cases} \tilde{X}_{\sigma(i)\sigma(j)} & \text{if } \sigma \text{ is even,} \\ \tilde{X}_{\sigma(i)\sigma(j)} - C_{\sigma(i)\sigma(j)} & \text{if } \sigma \text{ is odd.} \end{cases}$$

Also, the kernel of the Lie algebra epimorphism $\pi : \mathfrak{g}_{\boxtimes} \rightarrow \mathfrak{g}_{\boxtimes}$ is spanned by $\{C_p : p \in P\}$, and the C_p 's are fixed by the elements of Klein's 4 group. Hence, as in (2.1),

$$\mathfrak{g}_{\boxtimes} = \mathfrak{t} \oplus (\mathfrak{g}_{\boxtimes})_0 \oplus (\mathfrak{g}_{\boxtimes})_1 \oplus (\mathfrak{g}_{\boxtimes})_2,$$

where $\mathfrak{t} = \text{span}\{C_p : p \in P\}$, and the restriction $\pi|_{(\mathfrak{g}_{\boxtimes})_i} : (\mathfrak{g}_{\boxtimes})_i \rightarrow (\mathfrak{g}_{\boxtimes})_i$ is an isomorphism for any $i = 0, 1, 2$, where $(\mathfrak{g}_{\boxtimes})_i = \Psi^{-1}(\mathfrak{g}_i)$. It follows easily from here that the involution and the binary multiplication of the normal Lie related triple algebra associated to this S_4 -action on \mathfrak{g}_{\boxtimes} coincide (up to isomorphism) with the ones already considered for \mathfrak{g}_{\boxtimes} (actually for $\mathfrak{g} = \Psi(\mathfrak{g}_{\boxtimes})$). Also, the elements (compare to (1.8))

$$\begin{aligned}\hat{u}_0 &= \frac{1}{4}(Y_{02} + Y_{31}) = \frac{1}{4}(\tilde{X}_{02} + \tilde{X}_{31} - C_{02}), \\ \hat{u}_1 &= \frac{1}{4}(Y_{03} + Y_{12}) = \frac{1}{4}(\tilde{X}_{03} + \tilde{X}_{12} - C_{03}), \\ \hat{u}_2 &= \frac{1}{4}(Y_{01} + Y_{23}) = \frac{1}{4}(\tilde{X}_{01} + \tilde{X}_{23} - C_{01}),\end{aligned}$$

project onto the generators $\Psi^{-1}(u_0)$, $\Psi^{-1}(u_1)$ and $\Psi^{-1}(u_2)$ of \mathfrak{g}_{\boxtimes} (Theorem 1.9 and Corollary 1.18), and hence, since \mathfrak{g}_{\boxtimes} is perfect, they are generators of \mathfrak{g}_{\boxtimes} as a Lie algebra over k .

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